The Ray Method for the deflection of a Floating Flexible Platform in Short waves.

A.J. Hermans

Ray Method for the deflection of an inhomogeneous FFP

Ray hypothesis:

For propagating and evanescent modes (e.g. for deep water 3) we obtain:

- Equations for the phase and amplitude functions for inhomogeneous thin plate on water with depth (in)finite.
- Insufficient (initial) boundary conditions (2).

Canonical Problem ??

- Homogeneous half-plane problem
- Exact solution by means of differential-integral formulation.

Formulation

Dimensionless and harmonic $\Phi(\vec{x},t) = \phi(\vec{x})\ \mathrm{e}^{-\mathrm{i}\omega t}$:

$$\vec{x}' = \frac{\vec{x}}{L},\ h' = \frac{h}{L},\ K = \frac{\omega^2 L}{g},\ \mu = \frac{m\omega^2}{\rho g},\ \mathcal{D} = \frac{DK^4}{L^4 \rho g}$$

$$\Delta\Phi = 0 \quad \text{in the fluid,}$$

$$-K\phi + \frac{\partial\Phi}{\partial z} = 0 \quad \text{at } z = 0 \text{ and } (x,y) \in \mathcal{F}$$

$$\left\{ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \left(\frac{\mathcal{D}(x,y)}{K^4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right) + \frac{m(x,y)}{\rho g}\frac{\partial^2}{\partial t^2} + 1 \right\} \frac{\partial\phi}{\partial z} - K\phi = 0$$

$$\text{at } z = 0 \text{ and } (x,y) \in \mathcal{P}$$

Edge conditions

Consider the case with $D(x,y)$ and its normal derivative constant along the edge.

$$\frac{\partial^2 w}{\partial n^2} + \nu \frac{\partial^2 w}{\partial s^2} = 0$$

and

$$\frac{\partial^3 w}{\partial n^3} + (2-\nu)\frac{\partial^3 w}{\partial n \partial s^2} = 0$$

where

$$w = \frac{\mathrm{i}}{\omega L}\phi_z$$

The incident field

In the water region we have:

$$\phi^{inc}(\vec{x}) = \frac{g\zeta_\infty}{\mathrm{i}\omega}\frac{\cosh(k_0(z+h))}{\cosh(k_0 h)}\exp\{\mathrm{i}k_0(x\cos\beta + y\sin\beta)\}$$

with for finite depth:

$$k_0 \tanh(k_0 h) = K$$

and infinite depth:

$$K = k_0 = \omega^2 L/g$$

Underneath the platform

For $K \gg 1$ *Ray Ansatz*:

Assume n-modes.
For $h = \infty$ each mode is written as:

$$\phi(\vec{x},K) = \sum_n \alpha_n(\vec{x},K)\ \mathrm{e}^{\mathrm{i}KS_n(\vec{x})}$$

with:

$$\alpha_n(\vec{x},K) = \sum_{j=0}^{N} \frac{\alpha_{n,j}(\vec{x})}{(\mathrm{i}K)^j} + o((\mathrm{i}K)^{-N})$$

The Laplace equation yields for $z \leq 0$:

$$O(K^2)\ :\ \ \nabla_3 S \cdot \nabla_3 S = 0,$$

and

$$O(K^1)\ :\ \ 2\nabla_3\alpha_0 \cdot \nabla_3 S + \alpha_0 \Delta_3 S = 0$$

At the platform

At $z = 0$:

$$O(K^1) : \left\{ \mathcal{D}(x,y)(S_x^2 + S_y^2)^2 - \mu(x,y) + 1 \right\} \mathrm{i} S_z = 1$$

and $O(K^0)$:

$$\alpha_0 \mathcal{D} \left[\frac{\partial}{\partial z}(S_x^2 + S_y^2)^2 + 2S_z \left\{ \frac{\partial}{\partial x} S_x (S_x^2 + S_y^2) + \frac{\partial}{\partial y} S_y (S_x^2 + S_y^2) \right\} \right] +$$

$$\alpha_{0z} \left\{ \mathcal{D}(x,y)(S_x^2 + S_y^2)^2 - \mu(x,y) + 1 \right\} +$$

$$(4\mathcal{D}\nabla\alpha_0 \cdot \nabla S + 2\alpha \nabla \mathcal{D} \cdot \nabla S) S_z (S_x^2 + S_y^2) = 0$$

At the platform

Combine the equations in the fluid and at the plate:

With $r = \mathrm{i}S_z$ we get

$$(\mathcal{D}(x,y)r^4 - \mu(x,y) + 1)r = 1$$

in the deep water case and

$$(\mathcal{D}(x,y)r^4 - \mu(x,y) + 1)r \tanh rh = 1$$

in the finite water depth case

combined with:

$$S_x^2 + S_y^2 = r^2$$

This resembles the *eikonal* equation in acoustics.

Characteristics

For constant values of μ:

$$\frac{\mathrm{d}x}{\mathrm{d}\sigma} = \mathcal{F} S_x \ , \ \frac{\mathrm{d}y}{\mathrm{d}\sigma} = \mathcal{F} S_y \ ,$$

$$\frac{\mathrm{d}S_x}{\mathrm{d}\sigma} = -\mathcal{D}_x r^5 \ , \ \frac{\mathrm{d}S_y}{\mathrm{d}\sigma} = -\mathcal{D}_y r^5 \ , \ \frac{\mathrm{d}S}{\mathrm{d}\sigma} = \mathcal{F} r^2$$

with:

$$\mathcal{F} = (5\mathcal{D}r^4 - \mu + 1)/r$$

The amplitude equation becomes along the characteristic:

$$\frac{\mathrm{d}\alpha_0}{\mathrm{d}\sigma} = -\alpha_0 M\{S\}$$

with:

$$M\{S\}(x,y) = \frac{\frac{\mathrm{d}r}{\mathrm{d}\sigma}\{16r^4\mathcal{D} - \frac{1}{r}\} - \mathcal{F}(S_{xx} + S_{yy})(4\mathcal{D}r^5 + 1) + 4r^4\frac{\partial\mathcal{D}}{\partial\sigma}}{8\mathcal{D}r^5 + 2}$$

Boundary conditions

So far so good.

Three modes, two boundary conditions (at $z = 0$).

End of Story ??

Canonical Problem

Differential-Integral formulation:

$$4\pi\left(1-\mu(x,y)+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\mathcal{D}(x,y)}{K^4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right)w(x,y)$$

$$+K\int_{\mathcal{P}}\mathcal{G}(x,y;\xi,\eta)\left(\mu(\xi,\eta)-\left(\frac{\partial^2}{\partial \xi^2}+\frac{\partial^2}{\partial \eta^2}\right)\frac{\mathcal{D}(\xi,\eta)}{K^4}\left(\frac{\partial^2}{\partial \xi^2}+\frac{\partial^2}{\partial \eta^2}\right)\right)$$

$$\times w(\xi,\eta)\,\mathrm{d}S=4\pi\zeta^\infty\exp\{\mathrm{i}k_0(x\cos\beta+y\sin\beta)\}$$

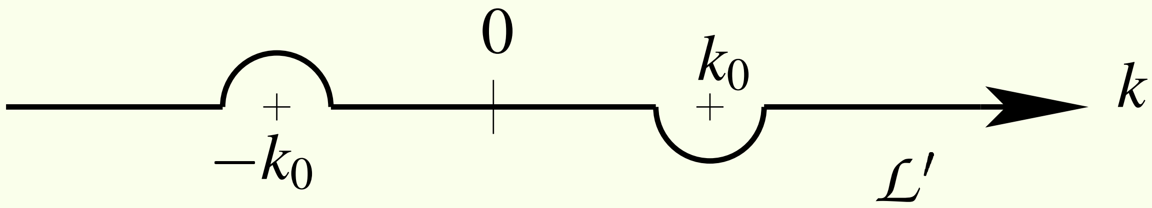
Green's Function

Green's function in 2-D at $z = \zeta = 0$ (1-D plaform):

$$\mathcal{G}(x;\xi) = -\int_{\mathcal{L}'} \frac{\cosh kh}{k \sinh kh - K \cosh kh}\, \mathrm{e}^{\mathrm{i}k(x-\xi)}\, \mathrm{d}k$$

Green's function in 3-D at $z = \zeta = 0$ (2-D plaform):

$$\mathcal{G}(x,y;\xi,\eta) = -2\int_0^\infty \frac{k \cosh kh}{k \sinh kh - K \cosh kh} J_0(kR)\, \mathrm{d}k$$



Geometrical-Optics *Ansatz*

Homogeneous platform:

$$w(x,y) = \sum_n a_n \exp\{\mathrm{i}\kappa_n x + \mathrm{i}k_0 y \sin\beta\}$$

for

$$0 \leq \beta \leq \beta_{cr} < \frac{\pi}{2}$$

Geometrical-Optics

We obtain:

$$\sum_n a_n \left(\frac{\mathcal{D}}{K^4} {\kappa^{(n)}}^4 - (\mu - 1) \right) \mathrm{e}^{\mathrm{i}\kappa_n x} = \zeta_\infty \, \mathrm{e}^{\mathrm{i}k_0 x \cos\beta} +$$
$$+ \mathrm{i} \sum_n a_n \frac{K}{4\pi} \left(\frac{\mathcal{D}}{K^4} {\kappa^{(n)}}^4 - \mu \right) \int_{\mathcal{L}'} \frac{\cosh kh}{k \sinh kh - K \cosh kh}$$
$$\left(\frac{\mathrm{e}^{\mathrm{i}x\sqrt{k^2 - k_0^2 \sin^2\beta}}}{\sqrt{k^2 - k_0^2 \sin^2\beta} - \kappa_n} - \frac{\mathrm{e}^{-\mathrm{i}x\sqrt{k^2 - k_0^2 \sin^2\beta}}}{\sqrt{k^2 - k_0^2 \sin^2\beta} + \kappa_n} \right) \frac{k \, \mathrm{d}k}{\sqrt{k^2 - k_0^2 \sin^2\beta}}$$

with

$${\kappa^{(n)}}^2 = \kappa_n^2 + k_0^2 \sin^2\beta$$

The integral

For $\beta = 0$ we have:

$$\mathrm{i}\sum_n a_n \frac{K}{2\pi}\left(\frac{\mathcal{D}}{K^4}{\kappa^{(n)}}^4 - \mu\right)\int_{\mathcal{L}'} \frac{\cosh kh}{k\sinh kh - K\cosh kh}\left(\frac{\mathrm{e}^{\mathrm{i}xk}}{k-\kappa_n} - \frac{\mathrm{e}^{-\mathrm{i}xk}}{k+\kappa_n}\right)\,\mathrm{d}k$$

If $\Im\kappa_n > 0$ then contribution of first term,

If $\Im\kappa_n < 0$ then contribution of second term.

Dispersion Relation

We close the contour either in the upper or lower half-plane

This results in the dispersion relation for $\kappa^{(n)}$:

$$\left(\frac{\mathcal{D}}{K^4}\kappa^4 - \mu + 1\right)\kappa \tanh \kappa h = K$$

we take into account one real (travelling wave)

and $N+2$ complex solutions (evanescent waves)

$N+3$ modes and TWO boundary conditions

Boundary conditions

For the amplitudes we obtain at $x = 0$

$$-\sum_{n=1}^{3}(\kappa_n^2 + \nu k_0^2 \sin^2\beta)a_n = 0$$

and

$$-\mathrm{i}\sum_{n=1}^{3}\kappa_n(\kappa_n^2 + (2-\nu)k_0^2 \sin^2\beta)a_n = 0$$

$N+1$ missing relation

From the pole at $k = k_0$ we get a third relation:

$$\sum_{n=0}^{N+2} \frac{K\left(\frac{\mathcal{D}\kappa^{(n)^4}}{K^4} - \mu\right) k_0 a_n}{(K(1-Kh)+k_0^2 h)\cos\beta(\kappa_n - k_0\cos\beta)} + \zeta_\infty = 0$$

and for $i = 1, \cdots, N$:

$$\sum_{n=0}^{N+2} \frac{K\left(\frac{\mathcal{D}\kappa^{(n)^4}}{K^4} - \mu\right) k_i a_n}{(K(1-Kh)+k_i^2 h)\sqrt{k_i^2 - k_0^2\sin^2\beta}\left(\kappa_n - \sqrt{k_i^2 - k_0^2\, sin^2\beta}\right)} = 0$$

Exact solution for a Strip

The same approach for the Strip with width L:

$$w(x,y) = \sum_{n=0}^{N} \left(a_n \exp\{\mathrm{i}\kappa_n x\} + b_n \exp\{-\mathrm{i}\kappa_n (x-L)\}\right) \exp\{\mathrm{i}k_0 y \sin\beta\}$$

Moving Rigid Strip ($\beta = 0$)

The equations of motion give two equations

$$\sum_{n=1}^{N}\left[\frac{a_n+b_n}{\mathrm{i}\kappa_n}\left(\mathrm{e}^{\mathrm{i}\kappa_n l}-1\right)\right]+\left[\frac{KM}{\rho}-l\right]\frac{w}{\zeta^{\infty}}=0$$

and

$$\sum_{n=1}^{N}(a_n-b_n)\left[\frac{1}{\kappa_n^2}\left(\mathrm{e}^{\mathrm{i}\kappa_n l}-1\right)+\frac{1}{\mathrm{i}\kappa_n}\left(\mathrm{e}^{\mathrm{i}\kappa_n l}+1\right)\right]+\left[\frac{KI}{\rho}-\frac{l^3}{12}\right]\frac{\theta}{\zeta^{\infty}}=0$$

where w and θ are the heave and pitch amplitude.
The pole analysis leads to $2N$ equations:

Moving Rigid Strip

$$k_0 \sum_{n=1}^{N} \left[\frac{a_n}{\kappa_n - k_0} - \frac{b_n \, \mathrm{e}^{\mathrm{i}\kappa_n l}}{\kappa_n + k_0} \right] + \frac{1}{\zeta^\infty} \left[w + \theta \left(\frac{1}{\mathrm{i}k_0} - \frac{l}{2} \right) \right] = Kh - 1 - \frac{k_0^2 h}{K}$$

and

$$k_0 \sum_{n=1}^{N} \left[\frac{-a_n \, \mathrm{e}^{\mathrm{i}\kappa_n l}}{\kappa_n + k_0} + \frac{b_n}{\kappa_n - k_0} \right] + \frac{1}{\zeta^\infty} \left[w - \theta \left(\frac{1}{\mathrm{i}k_0} - \frac{l}{2} \right) \right] \mathrm{e}^{\mathrm{i}k_0 l} = 0$$

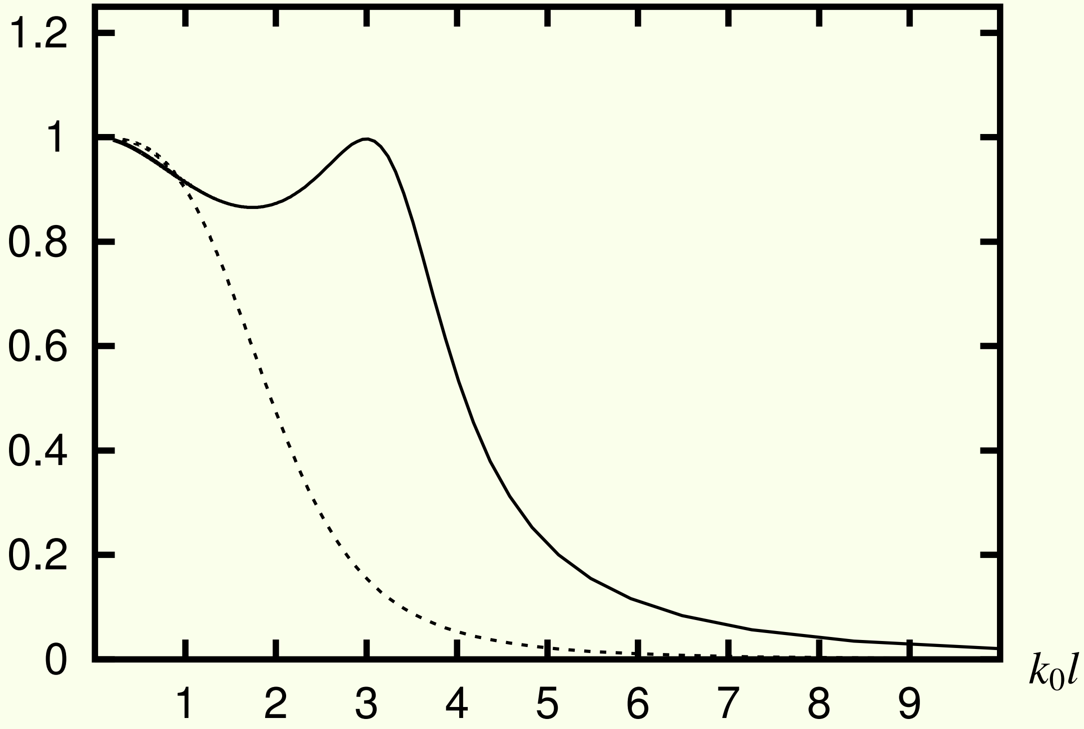
For $i = 1, \cdots, N-1$ we obtain:

$$k_i \sum_{n=1}^{N} \left[\frac{a_n}{\kappa_n - k_i} - \frac{b_n \, \mathrm{e}^{\mathrm{i}\kappa_n l}}{\kappa_n + k_i} \right] + \frac{1}{\zeta^\infty} \left[w + \theta \left(\frac{1}{\mathrm{i}k_i} - \frac{l}{2} \right) \right] = 0$$

and

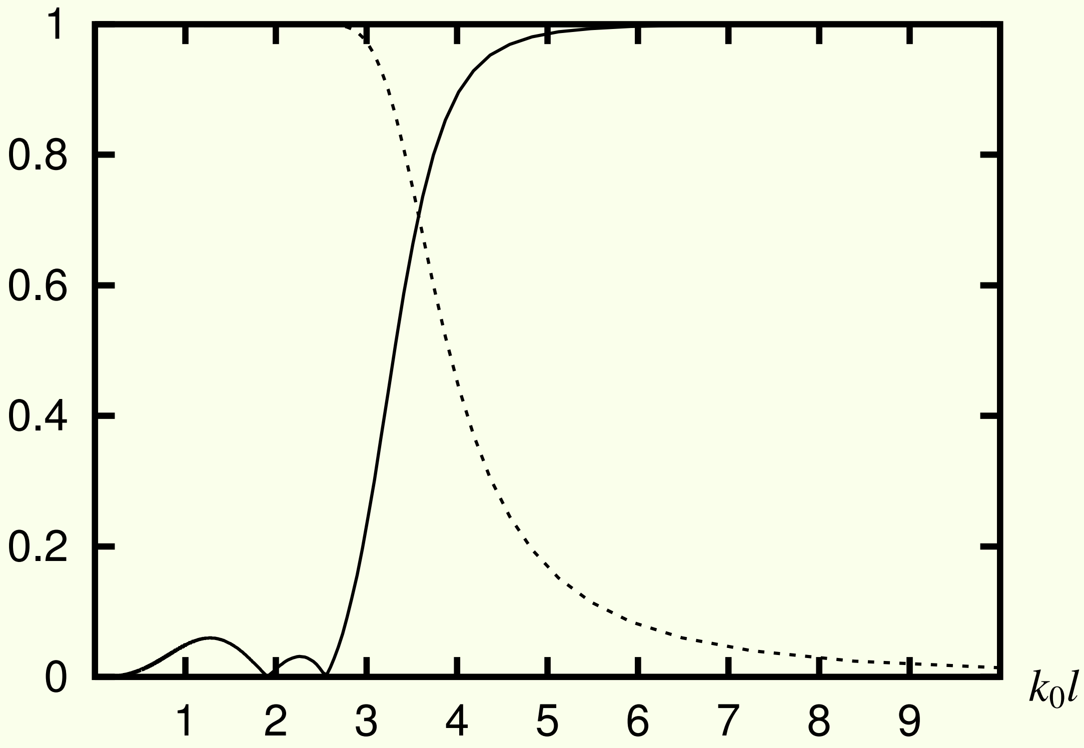
$$k_i \sum_{n=1}^{N} \left[\frac{-a_n \, \mathrm{e}^{\mathrm{i}\kappa_n l}}{\kappa_n + k_i} + \frac{b_n}{\kappa_n - k_i} \right] + \frac{1}{\zeta^\infty} \left[w - \theta \left(\frac{1}{\mathrm{i}k_i} - \frac{l}{2} \right) \right] \mathrm{e}^{\mathrm{i}k_i l} = 0$$

Heave and Pitch amplitudes



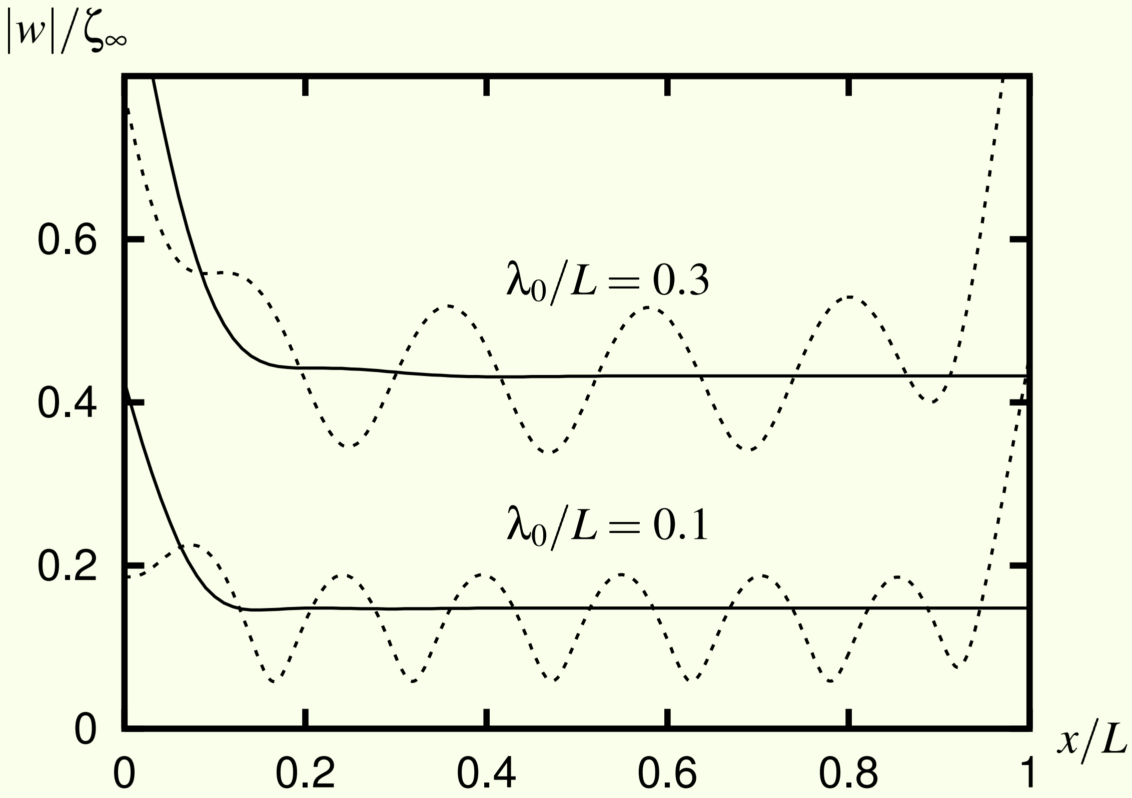
Amplitude of the heave, ___ w/ζ^∞ and pitch motion, ... $\theta/(k_0\zeta^\infty)$

Reflection and Transmission coefficients



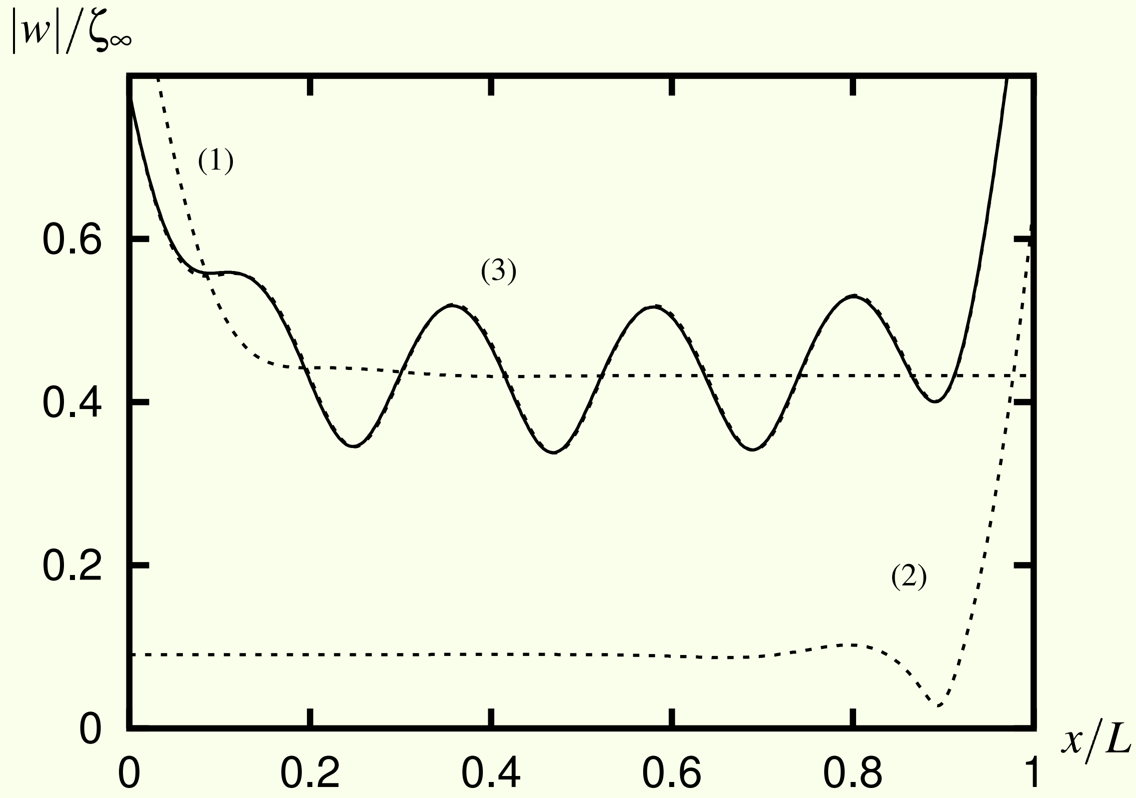
___ reflection and . . . transmission coefficient.

Flexible half-plane and strip



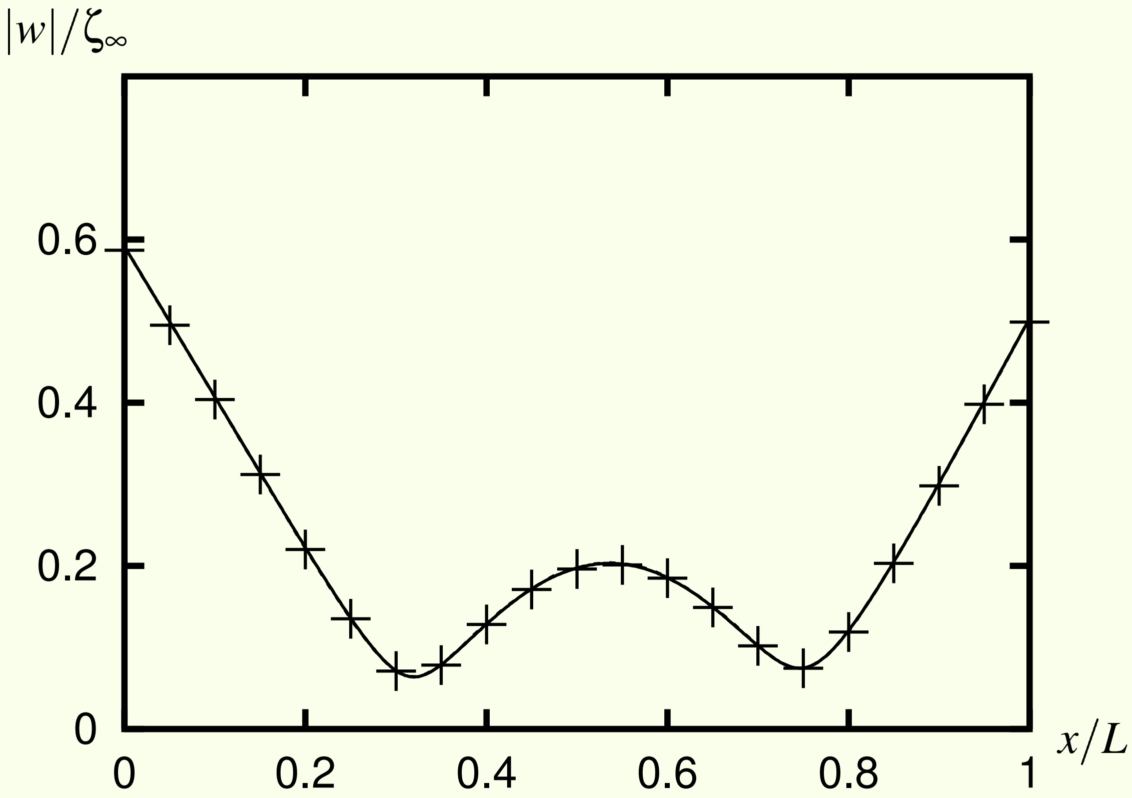
Asymptotic results for finite and semi-infinite (dotted) platform.Example with parameters $D/\rho g = 1.23 * 10^{-5} * L^4$, $h/L = 1/3$ and $\beta = 0$

Multiple reflection



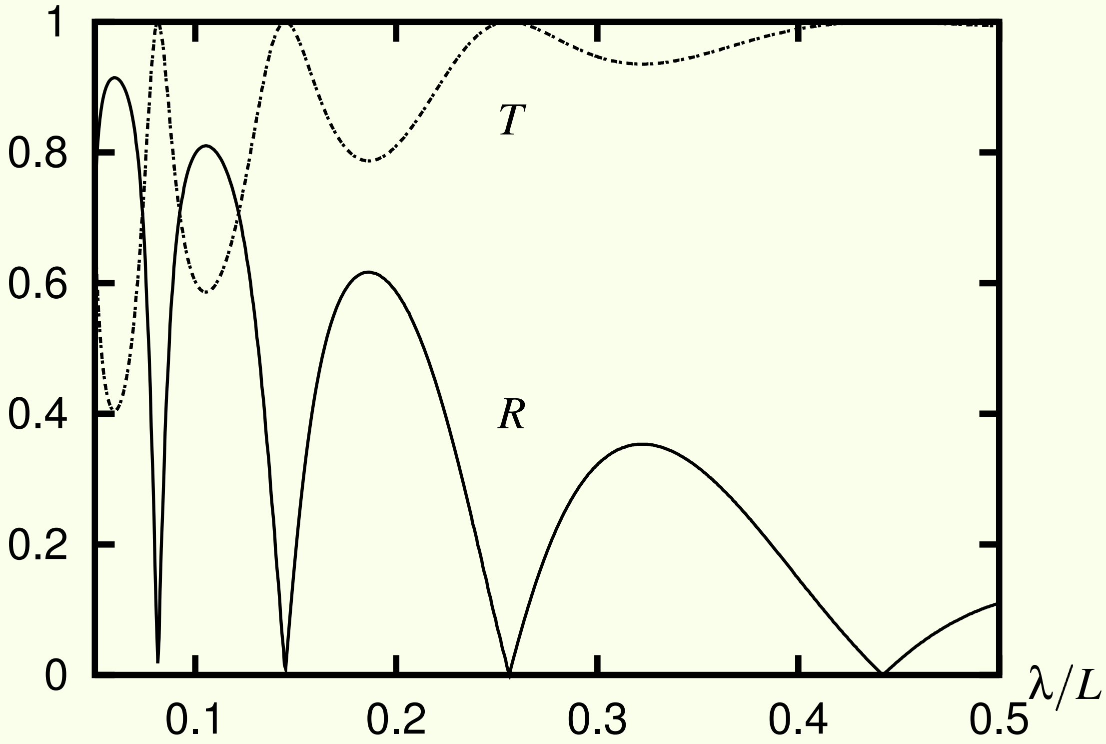
Multiple reflection $\lambda_0/L = 0.3$,

Comparison with Takagi



Comparison with the asymptotic results of Takagi for $\lambda_0/L = 0.278$ and $D/\rho g = 1.74 * 10^{-3} * L^4$, $h/L \approx 1/3$ and $\beta = 0$.

Reflection and Transmission Coefficient



Reflection and transmission coefficients $D/\rho g = 1.23 * 10^{-5} * L^4$, $h/L = 1/3$ and $\beta = 0$.

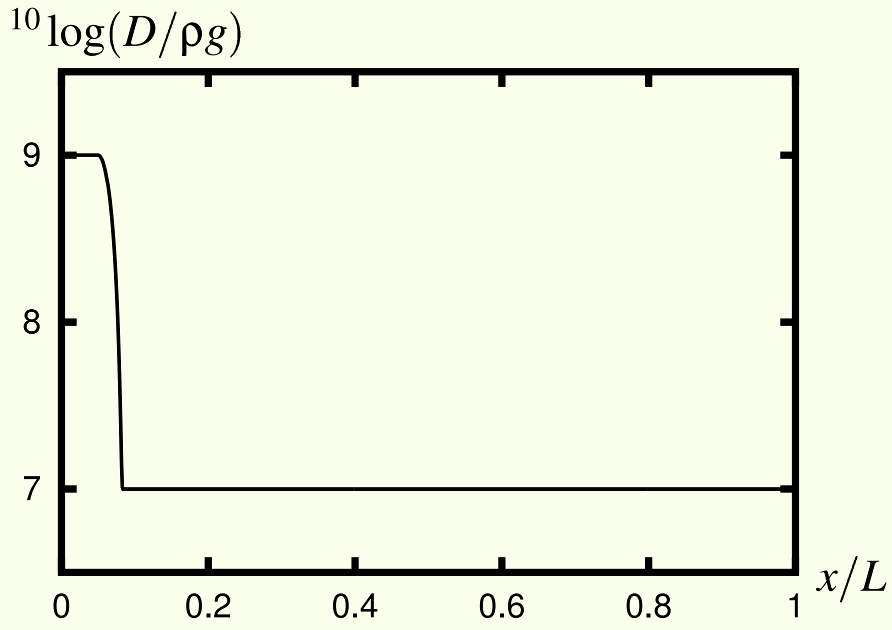
Application of the Ray Method

Inhomogeneous 2-D case with $D/(\rho g) = d(x)$:

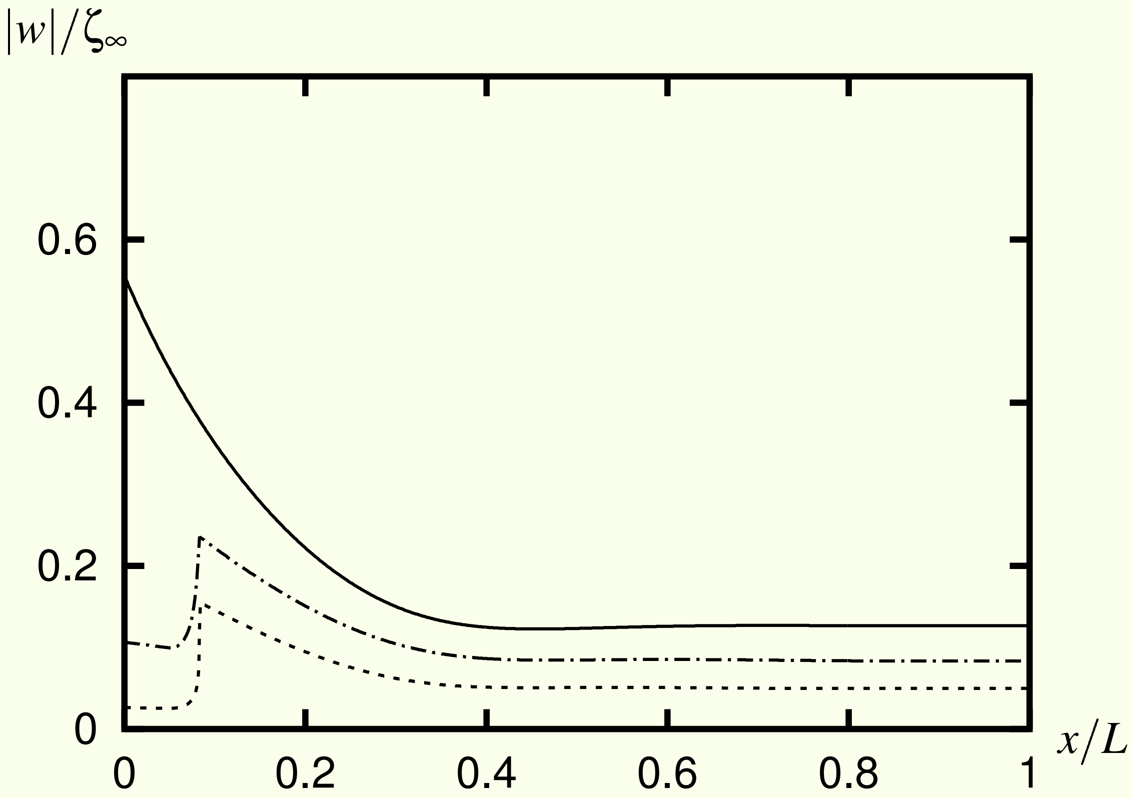
$$d = d^{(1)} = 10^9 \quad \text{for } 0 \leq x < 15 \text{ m}$$

$$d = \{d^{(1)} + d^{(2)} + (d^{(1)} - d^{(2)})\cos((x-15)\pi/10)\}/2 \quad \text{for } 15 \leq x < 25 \text{ m}$$

$$d = d^{(2)} = 10^7 \quad \text{for } 25 \leq x < \infty$$

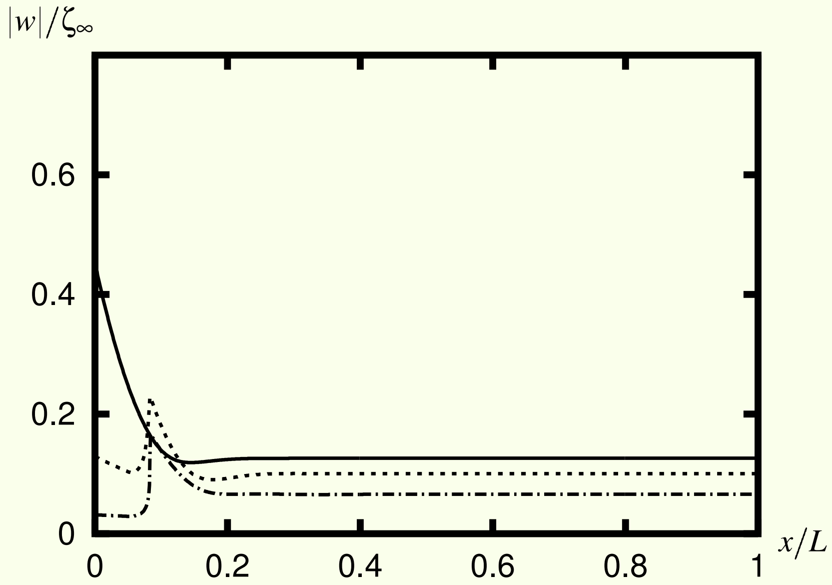


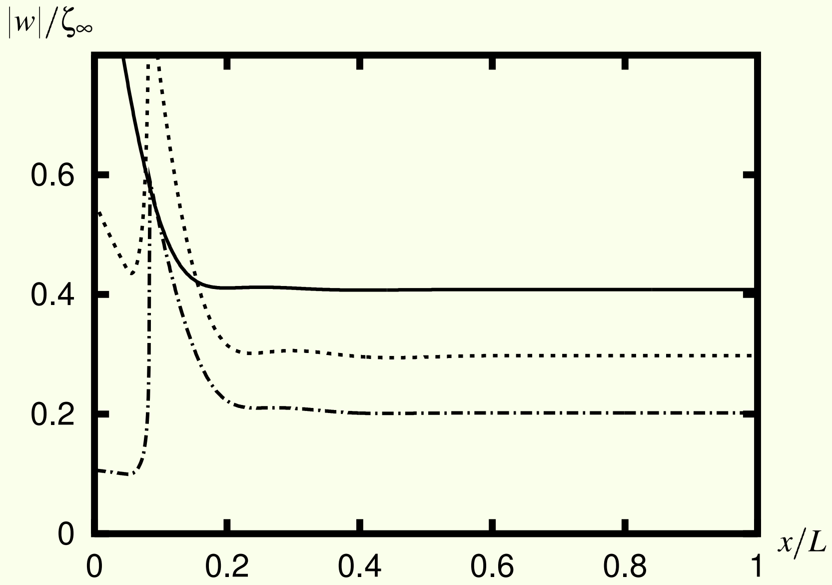
Semi infinite plate



Amplitude of deflection for $d = 10^7$ m^4 (top), and $d^{(1)} 10^9$ m^4 (middle), 10^{11} m^4 (bottom) resp. with $d^{(2)} = 10^7$ m^4, $\lambda_0/L = 0.3$ (with $L = 300$ m)

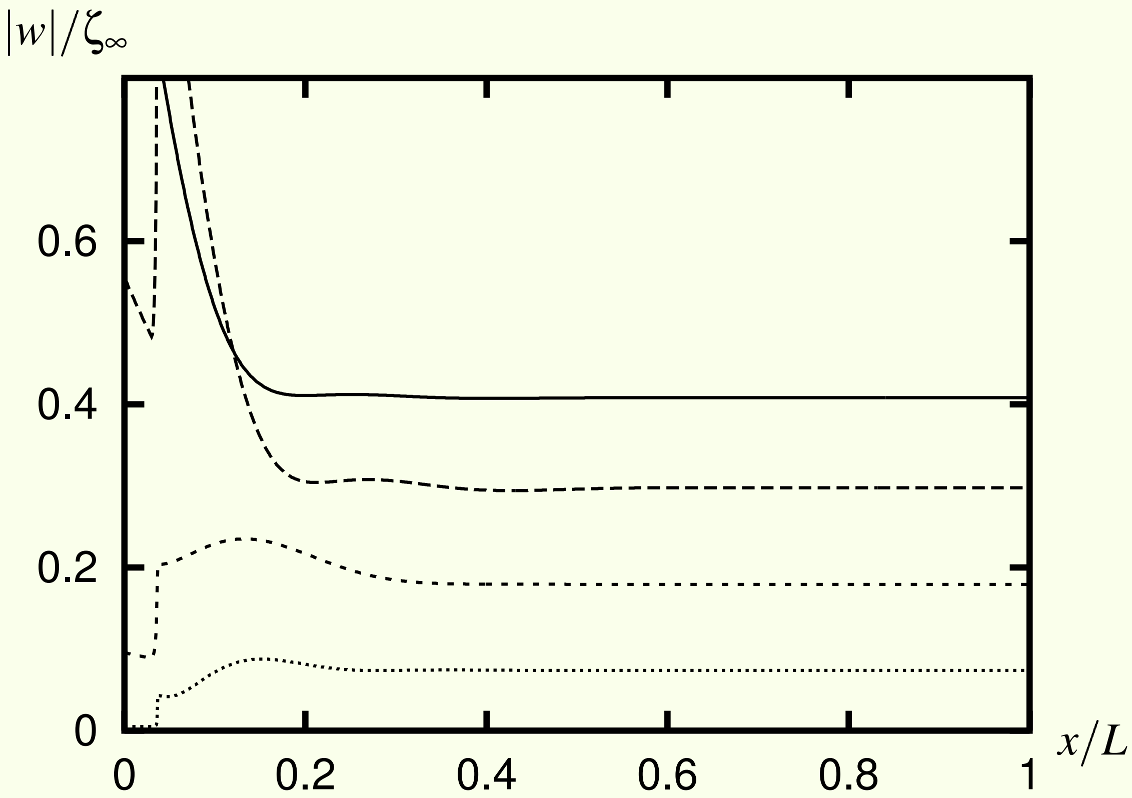
Variation of wave-length





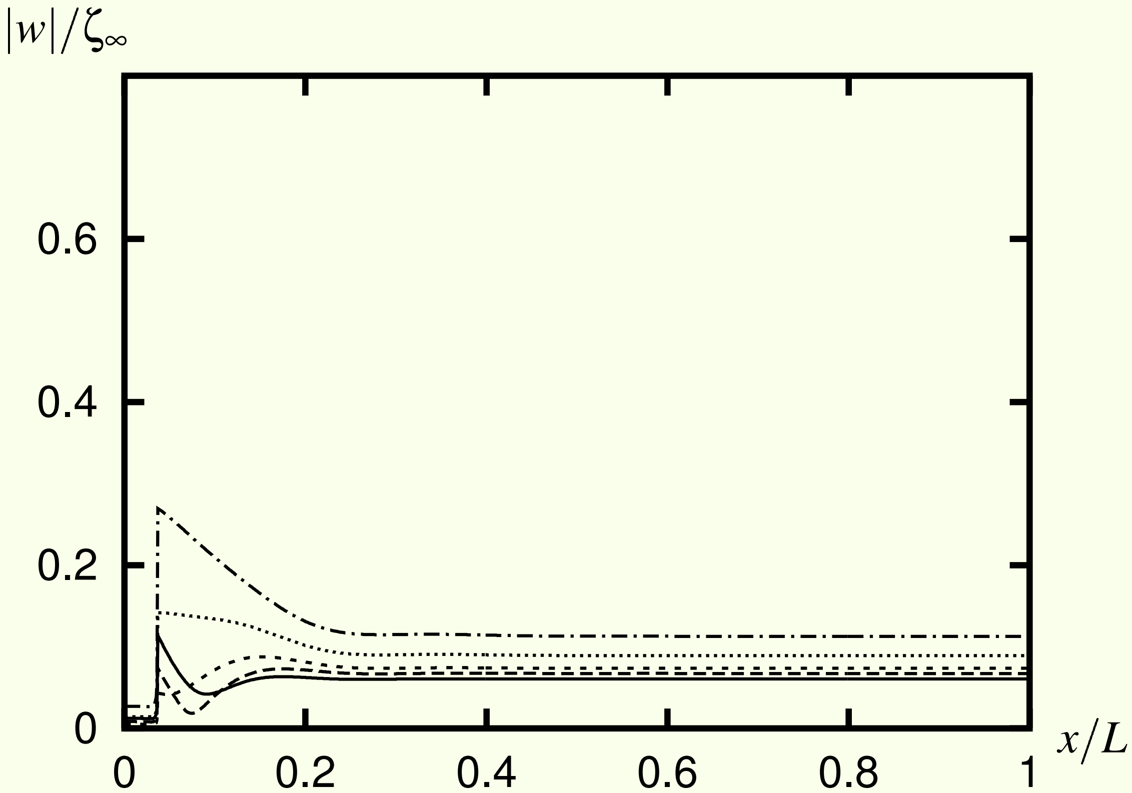
Amplitude of deflection for $d = 10^5$ m^4 (top), and $d^{(1)} = 10^7$ m^4 (middle), 10^9 m^4 (bottom) resp. with $d^{(2)} = 10^5$ m^4, $\lambda_0/L = 0.1$ (left) and $\lambda_0/L = 0.3$ (right) for a semi infinite plate ($L = 300$ m).

Extra weight at the edge



top to bottom $c = 0, d = 10^5$ m^4; $c = 0, d^{(1)} = 10^7$ m^4 and $d^{(2)} = 10^5$ m^4; $c = 2, d = 10^5$ m^4; $c = 2, d^{(1)} = 10^7$ m^4 and $d^{(2)} = 10^5$ m^4.

Variation of wave-length



top to bottom $c = 6, d^{(1)} = 10^{10}\ \mathrm{m}^4,\ d^{(2)} = 10^5\ \mathrm{m}^4,\ \lambda_0/L = 0.5, \cdots, 0.1\ \mathrm{m}$