Light diffusion in stochastically perturbed media

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In a medium such as biological tissue, the optical parameters may vary widely within a small volume. These small-scale variations cause variations in the light dose. A method is presented that quantifies (in the diffusion approximation) the average value and the standard deviation of the light fluence rate in a medium with stochastic optical parameters. Four optical parameters (μ_a, μ_s, g, and μ_s') are modeled separately at each point in the medium as samples of correlated distributions. We find that the mean value differs only slightly from the fluence rate calculated with the average optical parameters. When the standard deviation of the optical parameter is ~30%, the standard deviation of the stochastic fluence rate is of the same order of magnitude as the average fluence rate itself. Relative to the average value of the fluence rate, the standard deviation increases steadily with distance from the source: the fluence rate is more noisy deep in the medium.

INTRODUCTION

Traditionally, light distributions in scattering media are calculated with the use of deterministic optical parameters. These optical parameters are measured many times, and the average values are used for the calculations (see, e.g., Ref. 1). In that way, the media are assumed to be homogeneous on a small scale. In reality, only a few are. For example, in biological tissues the absorption coefficient varies widely within each cell. If one calculates light fluence rates in an inhomogeneous medium deterministically (using only the average values of the optical parameters), one also needs an indication of the range over which the actual (stochastic) fluence rates vary.

In this paper, the description of light transport is simplified by use of the diffusion approximation (P_1 approximation). In highly scattering media, diffusion theory describes the bulk of light transport and is accurate enough for describing the main stochastic effects. Furthermore, only one simple geometry is considered: an infinitely thick, stratified medium irradiated by a diffuse light beam. Although this one-dimensional geometry is in most cases not a realistic one, results for this geometry give an indication of the effects in other (three-dimensional) geometries.

A method is derived for calculating the expected value of the stochastic light fluence rate. If the expected value differs significantly from the deterministic fluence rate then a deterministic calculation is biased and inaccurate as is the case, for example, in Ref. 2. Furthermore, to quantify the variations of the stochastic fluence rate, we derived a method for calculating the standard deviation of the light fluence rate. The results are given for a typical biological tissue during a medical laser treatment.

METHOD

Diffusion Theory

To study the effect of stochastic optical parameters on the light fluence rate, a one-dimensional tissue geometry is chosen, as shown in Fig. 1. An infinitely wide and infinitely thick medium is irradiated by a diffuse light source (by diffuse we mean uniformly distributed in angle). Optically, the medium can be characterized by the following parameters: the absorption coefficient μ_a and the scattering coefficient μ_s (both in inverse centimeters); the mean cosine of the scattering angle g; and n, the index of refraction of the medium as compared with that of the surroundings. Derived optical parameters, used in diffusion theory, are the reduced scattering coefficient $\mu_s' = \mu_s(1 - g)$, and the reduced total attenuation coefficient $\mu_{tr} = \mu_a + \mu_s'$. In this paper, μ_a, μ_s, g, and μ_s' are considered stochastic parameters, one at a time.

Light transport in a highly scattering medium can be well approximated by diffusion theory.[3,4] In the geometry of Fig. 1, the governing equations for the light fluence rate $\Psi(z)$ and the light flux $F(z)$ (both in watts per square centimeter) are given by[5]

$$\frac{\mathrm{d}}{\mathrm{d}z}\Psi(z) + 3\mu_{tr}(z)F(z) = 0, \tag{1a}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}F(z) + \mu_a(z)\Psi(z) = 0, \tag{1b}$$

$$\Psi(0) + 2AF(0) = F_i, \tag{1c}$$

$$\lim_{z\to\infty}\Psi(z) = 0, \tag{1d}$$

where A and F_i account for internal reflections at the boundary surface[7]:

$$A = \frac{1 + R2_m}{1 - R1_m}, \tag{2}$$

$$F_i = 4F_{\mathrm{inc}}\frac{1 - R1_s}{1 - R1_m}. \tag{3}$$

Here F_{inc} is the incident flux. For each i, Ri is a different moment of the Fresnel reflection function for unpolarized light, $R(\mu)$. Subscript s refers to reflection of light in the surroundings of the medium at the surface of the medium. Subscript m refers to reflection of light in the medium at the other side of the same surface. When the index of refraction of the tissue and that of the surroundings match

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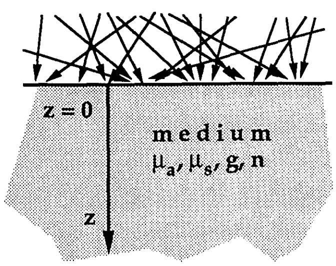

Fig. 1. Geometry: a one-dimensional medium of infinite thickness is irradiated with a diffuse light source with a total flux F_{inc}.

(i.e., $n = 1$), the terms Ri are zero. Then in Eq. (1c), factor A equals 1 and F_i equals $4F_{inc}$. When $n \neq 1$, the definition of Ri is

$$Ri = (i + 1)\int_0^1 \mu^i R(\mu)\mathrm{d}\mu, \tag{4}$$

where $R(\mu)$ is the Fresnel reflection as a function of μ, the cosine of the polar angle. The Ri's should be obtained by numerical integration.[8] For a medium with an index of refraction of 1.4, which is surrounded by air, we find that $A = 2.948$ and $F_i = 7.838\ F_{inc}$.

With use of an integrating sphere, the total of the light reflected by a sample, RT, can be measured (in watts per square centimeter). Within diffusion theory this quantity is a function of the fluence rate and the flux at surface $z = 0$ and can be calculated from

$$RT = \frac{(1 - R1_m)}{4}\Psi(0) - \frac{(1 - R2_m)}{2}F(0) + R1_s F_{inc}. \tag{5}$$

The last term in Eq. (5) describes the incident light directly reflected by the surface. With use of the boundary condition at $z = 0$ [Eq. (1c)], Eq. (5) can be simplified to

$$RT = \frac{\Psi(0)}{2A} + \frac{(-1 + 2R1_s + R2_m)}{(1 + R2_m)}F_{inc}. \tag{6}$$

For a medium with an index of refraction of 1.4 in air we obtain $RT = \Psi(0)/5.895 - 0.330F_{inc}$.

The solution of the diffusion equations, Eqs. (1), in a deterministic medium is straightforward. In what follows, deterministic parameters and quantities are given the subscript 0. The fluence rate and flux in a nonstochastic medium are given by

$$\Psi_0(z) = \frac{k_0 F_i}{k_0 + 2\mu_{a0}A}\exp(-k_0 z), \tag{7a}$$

$$F_0(z) = \frac{\mu_{a0} F_i}{k_0 + 2\mu_{a0}A}\exp(-k_0 z), \tag{7b}$$

where

$$k_0 = (3\mu_{a0}\mu_{tr0})^{1/2}. \tag{8}$$

Stochastic Medium

We now consider a medium with stochastic optical parameters. In such a medium the fluence rate is also a sample from a distribution and has an expected value and a standard deviation.

We consider a medium in which one of the optical parameters is sampled from a statistical distribution at each depth z. The one parameter that is varied is denoted μ. Parameter μ is μ_a, μ_s, g or the combination $\mu_s' = \mu_s(1 - g)$. We assume μ to be the sum of a constant average value, μ_0, plus a statistical disturbance, $\mu_1(z)$. Variation $\mu_1(z)$ is small compared with μ_0. The correlation between μ_1 distributions at two different depths is assumed to decrease exponentially with distance.[2,9–11]

$$\mu(z) \equiv \mu_0 + \mu_1(z), \tag{9a}$$

$$\mu_0 = \text{const.}, \tag{9b}$$

$$\langle \mu_1(z) \rangle = 0, \tag{9c}$$

$$\langle \mu_1(z)\mu_1(w) \rangle \equiv \alpha\sigma^2 \exp(-\alpha|z - w|), \tag{9d}$$

where $\langle f(\mu) \rangle$ signifies the expected value (average value) of f, i.e., $f(\mu)$ multiplied by the probability function of μ and integrated over all possible values of μ. Factor $\alpha\sigma^2$ in the correlation function determines the magnitude of the stochastic variations. For example, the absorption coefficient can be considered a stochastic variable. Then $\mu = \mu_a$, $\mu_a(z) = \mu_{a0} + \mu_{1a}(z)$, and $\sigma = \sigma_a$, while the scattering parameters are constant: $\mu_s(z) = \mu_{s0}$ and $g(z) = g_0$.

Parameter α in Eq. (9d) determines how strongly the μ_1 distributions at different depths correlate. In Fig. 2 some examples of μ_1 distributions are shown for different values of α.[12] If α is small, the μ_1 sample at one depth is close to the μ_1 samples at depths nearby. If α equals 0, the μ_1 samples are all the same and the medium is homogeneous with one (unknown) μ_1 value everywhere. For larger values of α, the distributions at different depths are less correlated. Distributions of $\mu_1(z)$ that are totally uncorrelated are called white noise in signal processing.[10,11] For white noise, one can obtain in two ways the results that follow: either by considering the limit for $\alpha \to \infty$ after using Eq. (9d) or by using a δ-Dirac function instead: $\langle \mu_1(z)\mu_1(w) \rangle \equiv \sigma^2\delta(z - w)$. Because most realistic media are not strongly correlated, the α values considered in this paper are limited to the larger values.

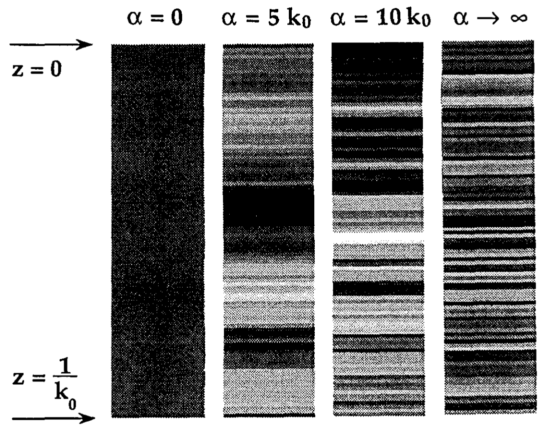

Fig. 2. Examples of media with stochastic optical parameters with different correlation functions. A dark pattern means a high μ_1 value. The probability distributions for $\mu_1(z)$ are the same at each depth, but, going from right to left, they are increasingly correlated among themselves. For the smaller α values a μ_1 sample has a high probability of being close to the μ_1 samples near it. For $\alpha = 0$, one μ_1 value is sampled and all other μ_1 values are the same. Most biological tissues can be considered uncorrelated: $\alpha \to \infty$; i.e., $\mu_1(z)$ is white noise.

Table 1. Quantities in Eqs. (10) and (15) for the Different Stochastic Parameters μ[a]

μ	P_a	P_b	x
μ_a	0	−1	$\dfrac{\sigma_a}{\mu_{a0}}$
μ_s	$-3(1 - g_0)$	0	$\dfrac{\mu_{s0}'}{\mu_{tr0}}\dfrac{\sigma_s}{\mu_{s0}}$
g	$3\mu_{s0}$	0	$\dfrac{\mu_{s0}'}{\mu_{tr0}}\dfrac{\sigma_g}{1 - g_0}$
μ_s'	−3	0	$\dfrac{\sigma_s'}{\mu_{tr0}}$

[a]Parameter x is a relative standard deviation of each stochastic parameter μ and (in highly scattering media) roughly equals σ [Eqs. (9)] divided by the average value of the parameter μ_0.

Light Fluence Rate as a Stochastic Process

After assumption of Eqs. (9), the varied optical parameter μ equals μ_a, μ_s, g, or μ_s'. When Eq. (9a) for a particular μ is substituted into Eqs. (1), then the following system is obtained:

$$\frac{\mathrm{d}}{\mathrm{d}z}\Psi(z) + 3\mu_{tr0}F(z) = P_a\mu_1(z)F(z), \tag{10a}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}F(z) + \mu_{a0}\Psi(z) = P_b\mu_1(z)\Psi(z), \tag{10b}$$

$$\Psi(0) + 2AF(0) = F_i, \tag{10c}$$

$$\lim_{z\to\infty}\Psi(z) = 0. \tag{10d}$$

Again the parameters with subscript 0 are the deterministic average values. Parameters P_a and P_b, which are defined in Table 1, differentiate system (10) into distinct systems for each parameter μ. When μ is one of the scattering parameters μ_s, g, or μ_s', then P_b equals zero. When μ is the absorption coefficient μ_a, we approximate P_a by zero [since in a highly scattering medium $\mu_{1a}(z) < \mu_{a0} \ll \mu_{tr0}$]. So either $P_b = 0$ or $P_a = 0$. A further study of Eqs. (10a) and (10b) then yields that the results when μ is a scattering parameter equal the results when μ is the absorption coefficient after $\Psi(z)$ has been interchanged with $F(z)$ and $3\mu_{tr0}$ with μ_{a0}.

System (10) cannot be averaged immediately, because the first statistical moments $\langle\mu_1(z)F(z)\rangle$ and $\langle\mu_1(z)\Psi(z)\rangle$ are not known yet. To evaluate these terms, we calculate the formal solutions of Eqs. (10a) and (10b) while treating the right-hand sides as the inhomogeneous parts. The formal solutions contain integrals over the same combinations of $\mu_1(z)$ and either $F(z)$ or $\Psi(z)$. Then the formal solutions for $\Psi(z)$ and $F(z)$ are substituted into their own integral terms. Thus all that remain are higher-order combinations of $\mu_1(z)$ and $\Psi(z)$ or $F(z)$ that can be averaged with use of a closure assumption.[13,14]

In the following equations, the only cases considered are those in which μ is one of the scattering parameters, thus when $P_b = 0$. As stated above, when μ is the absorption coefficient the equations can be simply obtained from the ones shown. When one of the scattering parameters is varied, the formal solutions of Eqs. (10a) and (10b) are

$$\Psi(z) = \cosh(k_0z)\Psi(0) - \frac{3\mu_{tr0}}{k_0}\sinh(k_0z)F(0) + P_a\int_0^z \cosh[k_0(z-s)]F(s)\mu_1(s)\mathrm{d}s, \tag{11a}$$

$$F(z) = -\frac{k_0}{3\mu_{tr0}}\sinh(k_0z)\Psi(0) + \cosh(k_0z)F(0) - \frac{k_0}{3\mu_{tr0}}P_a\int_0^z \sinh[k_0(z-s)]F(s)\mu_1(s)\mathrm{d}s. \tag{11b}$$

These expressions cannot yet be averaged, because $\langle\mu_1(s)F(s)\rangle$, the most important statistical quantity, is not known. We therefore substitute Eq. (11b) itself into the integrands of Eqs. (11a) and (11b). For Eq. (11a) this results in

$$\begin{aligned}\Psi(z) = {}& \cosh(k_0z)\Psi(0) - \frac{3\mu_{tr0}}{k_0}\sinh(k_0z)F(0) \\ & - P_a\int_0^z \cosh[k_0(z-s)] \\ & \times\left[\frac{k_0}{3\mu_{tr0}}\sinh(k_0s)\Psi(0) - \cosh(k_0s)F(0)\right]\mu_1(s)\mathrm{d}s \\ & - \frac{k_0}{3\mu_{tr0}}P_a^2\int_0^z\int_0^s \cosh[k_0(z-s)]\sinh[k_0(s-t)] \\ & \times\mu_1(s)\mu_1(t)F(t)\mathrm{d}t\mathrm{d}s,\end{aligned} \tag{12}$$

and a similar expression for $F(z)$. Equation (12) could be expanded further in the same way. At this point, however, we take its average value and use a closure assumption to discard higher-order correlation functions. The closure assumption used is Bourret's assumption of local independence[13,14]:

$$\langle\mu_1(z)\mu_1(w)\chi(z)\rangle \approx \langle\mu_1(z)\mu_1(w)\rangle\langle\chi(z)\rangle, \tag{13}$$

where

$$\chi(z) = \Psi(z) \quad \text{or} \quad \chi(z) = F(z).$$

It is thus assumed that the μ_1's correlate much more strongly among themselves than with $\Psi(z)$ or with $F(z)$; cross correlations can be disregarded. According to Frisch,[9] Bourret's assumption is justified if σ [Eqs. (9)] is small, while α/k_0 [Eqs. (8) and (9)] is large. So the variations in the optical parameters have to be small, and the medium has to be little correlated.[15]

To tackle the first integral term in Eq. (12), one needs another closure assumption. Only for correlations between $\mu_1(z)$ and the fluence rate or the flux at $z = 0$ we use a lower-order Bourret approximation:

$$\langle\mu_1(z)\chi(0)\rangle \approx \langle\mu_1(z)\rangle\langle\chi(0)\rangle = 0, \tag{14}$$

where

$$\chi(0) = \Psi(0) \quad \text{or} \quad \chi(0) = F(0).$$

A physical justification for this approximation is that most of the light at $z = 0$ comes directly from the (constant) source or after traversing only a short distance through the medium. So at $z = 0$ the correlation of the light with the medium is small.[17]

Average Value of the Light Fluence Rate

In Table 1 we introduce parameter x as a relative standard deviation of the stochastic parameter μ. For example, if $\mu = \mu_s$ and $x = 30\%$ then the standard deviation of μ_s, σ_s, is almost 30% of the value of μ_s itself. The relative standard deviation x is scaled with factors that are close to 1, because in the problems considered, $\mu_{a0} \ll \mu_{s0}'$, so $\mu_{tr0} = \mu_{a0} + \mu_{s0}' \approx \mu_{s0}'$. Note that σ_g is taken relative to $(1 - g_0)$ rather than to g_0 itself.

If we take the expected value of Eq. (12) and use Bourret's assumptions, Eqs. (13) and (14) yield an integral equation for $\langle\Psi(z)\rangle$:

$$\begin{aligned}\langle\Psi(z)\rangle &= \cosh(k_0 z)\langle\Psi(0)\rangle - \frac{3\mu_{tr0}}{k_0}\sinh(k_0 z)\langle F(0)\rangle \\ &\quad - 3\mu_{tr0}k_0\alpha x^2 \int_0^z \int_0^s \cosh[k_0(z-s)] \\ &\quad \times \sinh[k_0(s-t)]\exp[-\alpha(s-t)]\langle F(t)\rangle \mathrm{d}t\mathrm{d}s. \qquad (15)\end{aligned}$$

The expression for $\langle F(z)\rangle$ is similar. As a next step, we reverse the order of integration in Eq. (15) and analytically integrate over s. After a Laplace transformation we obtain

$$\begin{aligned}L\{\langle\Psi(z)\rangle\} &= \frac{s\langle\Psi(0)\rangle - 3\mu_{tr0}\langle F(0)\rangle}{s^2 - k_0^2} \\ &\quad - \frac{3\mu_{tr0}k_0^2\alpha x^2 s}{(s^2 - k_0^2)[(s+\alpha)^2 - k_0^2]} L\{\langle F(z)\rangle\}, \\ L\{\langle F(z)\rangle\} &= \frac{-\mu_{a0}\langle\Psi(0)\rangle + s\langle F(0)\rangle}{s^2 - k_0^2} \\ &\quad + \frac{k_0^4\alpha x^2}{(s^2 - k_0^2)[(s+\alpha)^2 - k_0^2]} L\{\langle F(z)\rangle\}. \qquad (16)\end{aligned}$$

This system is solved and so, when $\mu = \mu_s$, $\mu = g$, or $\mu = \mu_s'$, the Laplace transforms of the average fluence rate and the average flux are given by

$$L\{\langle\Psi(z)\rangle\} = \frac{[(s+\alpha)^2 - k_0^2]s\langle\Psi(0)\rangle - 3\mu_{tr0}[(s+\alpha)^2 - k_0^2 + k_0^2\alpha x^2]\langle F(0)\rangle}{(s^2 - k_0^2)[(s+\alpha)^2 - k_0^2] - k_0^4\alpha x^2}, \qquad (17a)$$

$$L\{\langle F(z)\rangle\} = \frac{[(s+\alpha)^2 - k_0^2][-\mu_{a0}\langle\Psi(0)\rangle + s\langle F(0)\rangle]}{(s^2 - k_0^2)[(s+\alpha)^2 - k_0^2] - k_0^4\alpha x^2}. \qquad (17b)$$

By first averaging and then Laplace transforming Eq. (10b), we find a relation between $L\{\langle\Psi(z)\rangle\}$ and $L\{\langle F(z)\rangle\}$:

$$L\{\langle\Psi(z)\rangle\} = -\frac{1}{\mu_{a0}}(sL\{\langle F(z)\rangle\} - \langle F(0)\rangle), \qquad (18)$$

which relation is implicit in Eqs. (17). When $\mu = \mu_a$ is considered the statistical parameter, one can find $L\{\langle\Psi(z)\rangle\}$ and $L\{\langle F(z)\rangle\}$ from Eqs. (17) and (18) by changing $\Psi(z)$ into $F(z)$ and μ_{a0} into $3\mu_{tr0}$ and vice versa.

Functions $\langle\Psi(z)\rangle$ and $\langle F(z)\rangle$ themselves are found by inverse Laplace transforming Eqs. (17) and then applying the boundary conditions for $\langle\Psi(z)\rangle$ and $\langle F(z)\rangle$. The numerator of the expression for $L\{\langle\Psi(z)\rangle\}$ in Eq. (17a) is denoted $U_1(s)$, and the denominator is $U_2(s)$. First, using the computer algebra program MATHEMATICA, we obtain the roots of $U_2(s)$, denoted s_1, s_2, s_3, and s_4. After decomposition of the fractions in Eqs. (17a), the inverse Laplace transform can be applied. The resulting average fluence rate is

$$\langle\Psi(z)\rangle = \sum_{i=1}^{4} \frac{U_1(s_i)}{(\mathrm{d}/\mathrm{d}s)U_2(s_i)}\exp(s_i z). \qquad (19)$$

The values of the roots s_i show the necessity of the conditions for the Bourret approximation [relation (13)]. If the relative standard deviation x is too large, two roots are complex numbers giving an oscillating $\langle\Psi(z)\rangle$, which is physically unacceptable. If correlation parameter α is too small, two roots are positive numbers, so that $\langle\Psi(z)\rangle$ cannot vanish at $z \to \infty$.

The two unknowns remaining, $\langle\Psi(0)\rangle$ and $\langle F(0)\rangle$, are found from the two boundary conditions for $\langle\Psi\rangle$ and $\langle F\rangle$:

$$\langle\Psi(0)\rangle + 2A\langle F(0)\rangle = F_i, \qquad (20a)$$

$$\lim_{z\to\infty}\langle\Psi(x)\rangle = 0. \qquad (20b)$$

The conditions in Eqs. (20) are just the average values of the boundary conditions for Ψ and F [Eqs. (10c) and (10d)].[18] For α values that are large enough and x values that are small enough, the roots s_1, s_2, s_3, and s_4 of $U_2(s)$ are real, while three are negative and one, say, s_4, is positive. The value of s_4 is approximately $+k_0$. For compliance with the boundary condition for $z \to \infty$ [Eq. (20b)], the coefficient in Eq. (19) for s_4 is made to vanish, giving a second relation between $\langle\Psi(0)\rangle$ and $\langle F(0)\rangle$; $U_1(s_4) = 0$. The second-largest root is now the most important one in Eq. (19). Its value is approximately $-k_0$. Note that the exponent of the fluence rate in a deterministic medium [Eqs. (7)] is exactly $-k_0$.

If the $\mu_1(z)$ distributions at different depths are not correlated, i.e., $\mu_1(z)$ is white noise, one can find $L\{\langle\Psi(z)\rangle\}$ and $L\{\langle F(z)\rangle\}$ either by using a δ-Dirac function instead of Eqs. (9) as a correlation function or, equivalently, by taking in Eqs. (17) the limit for $\alpha \to \infty$. Then the expected value of the fluence rate does not depend on parameter x. When $\mu = \mu_s$, $\mu = g$, or $\mu = \mu_s'$ we have

$$L_{\alpha\to\infty}\{\langle\Psi(z)\rangle\} = \frac{s\langle\Psi(0)\rangle - 3\mu_{tr0}\langle F(0)\rangle}{s^2 - k_0^2}, \qquad (21a)$$

$$L_{\alpha\to\infty}\{\langle F(z)\rangle\} = \frac{-\mu_{a0}\langle\Psi(0)\rangle + s\langle F(0)\rangle}{s^2 - k_0^2}. \qquad (21b)$$

Solving Eqs. (21) yields $\langle\Psi(z)\rangle = \Psi_0(z)$, defined in Eq. (7a), and $\langle F(z)\rangle = F_0(z)$, defined in Eq. (7b). So when the medium is totally uncorrelated, the average values of the stochastic fluence rate and of the flux equal the fluence rate and the flux in a constant deterministic medium.

Standard Deviation of the Light Fluence Rate

The standard deviation of the fluence rate and of the flux are defined by

$$\begin{aligned}\sigma\{\Psi(z)\} &= [\langle\Psi^2(z)\rangle - \langle\Psi(z)\rangle^2]^{1/2}, \\ \sigma\{F(z)\} &= [\langle F^2(z)\rangle - \langle F(z)\rangle^2]^{1/2}. \qquad (22)\end{aligned}$$

The method used to calculate $\langle\Psi^2(z)\rangle$ and $\langle F^2(z)\rangle$ is similar to the one used above to obtain $\langle\Psi(z)\rangle$ and $\langle F(z)\rangle$. This time we start from a system of three differential equations for $\Psi^2(z)$, $\Psi(z)F(z)$, and $F^2(z)$. Equations (10a) and (10b) are multiplied either by $\Psi(z)$ or by $F(z)$, and the resulting system is given by

$$\frac{\mathrm{d}}{\mathrm{d}z}\Psi^2(z) + 6\mu_{tr0}\Psi(z)F(z) = 2P_a\mu_1(z)\Psi(z)F(z), \quad (23a)$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\Psi(z)F(z) + \mu_{a0}\Psi^2(z) + 3\mu_{tr0}F^2(z) = \mu_1(z)[P_b\Psi^2(z) + P_aF^2(z)], \quad (23b)$$

$$\frac{\mathrm{d}}{\mathrm{d}z}F^2(z) + 2\mu_{a0}\Psi(z)F(z) = 2P_b\mu_1(z)\Psi(z)F(z). \quad (23c)$$

This system is first solved formally while the right-hand-side terms are treated as sources, just as was done with the first statistical moments in Eqs. (11). Then the three formal solutions are substituted once into the integral terms, as for Eq. (12). The equations are averaged, and Bourret's assumptions are used [relations (13) and (14) for those higher-order moments]. The results now resemble Eq. (15). The order of integration is reversed in the double-integral terms, and one of the integrations is performed analytically. The system of three integral equations in $\langle\Psi^2(z)\rangle$, $\langle\Psi(z)F(z)\rangle$, and $\langle F^2(z)\rangle$ is Laplace transformed [as for Eqs. (16)], and then solved for $L\{\langle\Psi^2(z)\rangle\}$, $L\{\langle\Psi(z)F(z)\rangle\}$, and $L\{\langle F^2(z)\rangle\}$. When $\mu = \mu_s$, g, or μ_s', we obtain

$$L\{\langle\Psi^2(z)\rangle\} = \frac{f_1(s)\langle\Psi^2(0)\rangle + f_2(s)\langle\Psi(0)F(0)\rangle + f_3(s)\langle F^2(0)\rangle}{[(s+\alpha)^2 - 4k_0^2](s^2 - 4k_0^2)(s+\alpha)s - 4\alpha k_0^4x^2(2s+\alpha)^2},$$

$$L\{\langle F^2(z)\rangle\} = \frac{f_4(s)\langle\Psi^2(0)\rangle + f_5(s)\langle\Psi(0)F(0)\rangle + f_6(s)\langle F^2(0)\rangle}{[(s+\alpha)^2 - 4k_0^2](s^2 - 4k_0^2)(s+\alpha)s - 4\alpha k_0^4x^2(2s+\alpha)^2},$$

$$L\{\langle\Psi(z)F(z)\rangle\} = -\frac{1}{2\mu_{a0}}[sL\{\langle F^2(z)\rangle\} - \langle F^2(0)\rangle], \quad (24a)$$

where

$$\begin{aligned} f_1(s) &= [(s+\alpha)^2 - 4k_0^2](s+\alpha)(s^2 - 2k_0^2) - 4\alpha k_0^4x^2(2s+\alpha), \\ f_2(s) &= -6\mu_{tr0}\{[(s+\alpha)^2 - 4k_0^2](s+\alpha)s + 2\alpha k_0^2x^2(2s+\alpha)(s+\alpha)\}, \\ f_3(s) &= 18\mu_{tr0}^2\{[(s+\alpha)^2 - 4k_0^2](s+\alpha) + \alpha x^2(s^2 + \alpha s + 4k_0^2)(s+\alpha)\}, \\ f_4(s) &= 2\mu_{a0}^2[(s+\alpha)^2 - 4k_0^2](s+\alpha), \\ f_5(s) &= -2\mu_{a0}[(s+\alpha)^2 - 4k_0^2](s+\alpha)s, \\ f_6(s) &= [(s+\alpha)^2 - 4k_0^2](s+\alpha)(s^2 - 2k_0^2) - 4\alpha k_0^4x^2(2s+\alpha). \end{aligned} \quad (24b)$$

If $\mu = \mu_a$, the same equations as for Eqs. (24) apply, only the Ψ's and F's have to be interchanged, and simultaneously the factors μ_{a0} and $3\mu_{tr0}$ have to be interchanged. Inverse Laplace transformation of Eqs. (24) gives $\langle\Psi^2(z)\rangle$, $\langle\Psi(z)F(z)\rangle$, and $\langle F^2(z)\rangle$. To that end the six roots (labeled $s_j, j = 1\ldots6$) of the denominators of Eq. (24a) are determined again numerically. For sufficiently large values of α and for sufficiently small values of x, all roots are real valued. One root is positive (say, s_6), one is nearly zero but negative (say, s_5), one is approximately $-2k_0$, and the other three roots are smaller.[19]

Three parameters are still unknown: $\langle\Psi^2(0)\rangle$, $\langle\Psi(0)F(0)\rangle$, and $\langle F^2(0)\rangle$. They are used for compliance with the boundary conditions and with some physical conditions. First, a condition must be imposed on the solution: $\Psi(z)$ vanishes for $z \to \infty$ [Eq. (1d)]; thus $\Psi^2(z)$ vanishes also for $z \to \infty$, and consequently

$$\lim_{z\to\infty}\langle\Psi^2(z)\rangle = 0. \quad (25)$$

Second, Var$\{\Psi(z)\}$ and Var$\{F(z)\}$ [the squares of the standard deviations of Eqs. (22)] have to be nonnegative. So extra conditions for $\langle\Psi^2(z)\rangle$ and $\langle F^2(z)\rangle$ are

$$\langle\Psi^2(z)\rangle \geq \langle\Psi(z)\rangle^2, \qquad \langle F^2(z)\rangle \geq \langle F(z)\rangle^2 \quad \text{for all } z. \quad (26)$$

At $z = 0$, the exact boundary condition for $\langle\Psi^2(z)\rangle$, $\langle\Psi(z)F(z)\rangle$, and $\langle F^2(z)\rangle$ equals boundary condition (10c) squared and then averaged.[20] However, the solutions cannot comply with the exact boundary condition at $z = 0$, with the boundary condition for $z \to \infty$ [condition (25)] and with conditions (26) at the same time. We presume that this situation is caused by our having discarded the higher-order correlations between the light and the medium [expressions (13) and (14)]. Instead, $\langle\Psi^2(0)\rangle$, $\langle\Psi(0)F(0)\rangle$, and $\langle F^2(0)\rangle$ are chosen such that the solutions comply with conditions (25) and (26) and such that at the same time, the exact boundary condition at $z = 0$ is approximated as nearly as possible:

$$\mathrm{Min}\{\langle\Psi^2(0)\rangle + 4A\langle\Psi(0)F(0)\rangle + 4A^2\langle F^2(0)\rangle - F_i^2| \text{ given conditions (25) and (26)}\}. \quad (27)$$

Condition (25) requires that the coefficient of the term with the positive root s_6 vanish. Condition (26) requires the same for the term with the nearly zero root s_5.[21] The third degree of freedom is then used to minimize the difference with boundary condition (27).

If the medium is not correlated, the second statistical moments can be calculated again by taking system (24) in the limit for $\alpha \to \infty$. Contrary to $\langle\Psi(z)\rangle$ and $\langle F(z)\rangle$ for $\alpha \to \infty$, the second statistical moments do depend on parameter x. The Laplace transformed moments are for $\mu = \mu_s$, g, or μ_s':

$$\begin{aligned} L_{\alpha\to\infty}\{\langle\Psi^2(z)\rangle\} &= \frac{(s^2 - 2k_0^2)\langle\Psi^2(0)\rangle - 6\mu_{tr0}(s + 2k_0^2x^2)\langle\Psi(0)F(0)\rangle + 18\mu_{tr0}^2(1 + x^2s)\langle F^2(0)\rangle}{(s^2 - 4k_0^2)s - 4k_0^4x^2}, \\ L_{\alpha\to\infty}\{\langle\Psi(z)F(z)\rangle\} &= \frac{-\mu_{a0}s\langle\Psi^2(0)\rangle + s^2\langle\Psi(0)F(0)\rangle - 3\mu_{tr0}(s + 2k_0^2x^2)\langle F^2(0)\rangle}{(s^2 - 4k_0^2)s - 4k_0^4x^2}, \\ L_{\alpha\to\infty}\{\langle F^2(z)\rangle\} &= \frac{2\mu_{a0}\langle\Psi^2(0)\rangle - 2\mu_{a0}s\langle\Psi(0)F(0)\rangle + (s^2 - 2k_0^2)\langle F^2(0)\rangle}{(s^2 - 4k_0^2)s - 4k_0^4x^2}. \end{aligned} \quad (28)$$

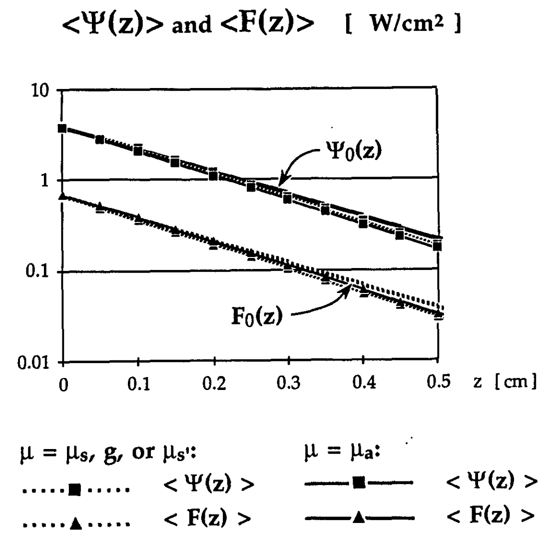

Fig. 3. Expected values of the light fluence rate and of the light flux in a medium with a stochastic optical parameter $\mu_{a0} = 1$/cm, $\mu_{s0} = 100$/cm, $g_0 = 0.9$, $\mu_{tr0} = 11$/cm, and $n = 1.4$. The medium is rather correlated: $\alpha = 5k_0 \approx 29$/cm. Parameter x equals 30%, so the standard deviation of the stochastic optical parameter μ is roughly 30% of μ_0. The average fluence rates $\langle\Psi(z)\rangle$ and the average fluxes $\langle F(z)\rangle$ differ only slightly from the deterministic fluence rate $\Psi_0(z)$ and the deterministic flux $F_0(z)$. Deterministic calculations with the average optical parameters can be considered unbiased.

Only when $\alpha \to \infty$, is boundary condition (27) minimized by taking $\langle\Psi^2(0)\rangle = \langle\Psi(0)\rangle^2$ (if $\mu = \mu_s$, g, or μ_s'), or $\langle F^2(0)\rangle = \langle F(0)\rangle^2$ (if $\mu = \mu_a$). The other two parameters at $z = 0$ are used to remove the two terms resulting from roots s_5 and s_6.

RESULTS

The characteristics of a stochastic light distribution are shown in a sample background medium with different variations in the optical parameters. The absorption coefficient is considered to be stochastic, $\mu = \mu_a$; or the scattering coefficient, $\mu = \mu_s$; or the mean cosine of the scattering angle, $\mu = g$; or the reduced scattering coefficient, $\mu = \mu_s'$. The amplitude of the variations, determined by parameter x, is varied in Figs. 4 and 5. Then the effect is shown on the standard deviation of the fluence rate when the correlation parameter α is varied. In Fig. 7 the main result of this paper is demonstrated: the relative standard deviation of the fluence rate increases deeper in the medium.

The sample medium is a typical biological tissue during a medical laser treatment. The average optical parameters are $\mu_{a0} = 1$/cm, $\mu_{s0} = 100$/cm, $g_0 = 0.9$, $\mu_{s0}' = 10$/cm, and $n = 1.4$. The medium is highly scattering, so diffusion theory is appropriate. The total incident flux is $F_{\text{inc}} = 1$ W/cm^2.

The relative standard deviation of the stochastic optical parameter x is either 10% or 30%. When $x = 30\%$, the standard deviation of the parameter is approximately 30% of the average value of the parameter: $\sigma_a = 0.3$/cm, $\sigma_s = 33$/cm, $\sigma_g = 0.033$, or $\sigma_s' = 3.3$/cm, when μ is μ_a, μ_s, g, or μ_s', respectively. For the same value of x, the stochastic fluence rates and fluxes for the cases $\mu = \mu_s$, $\mu = g$, and $\mu = \mu_s'$ are identical. Only if $\mu = \mu_a$ is the stochastic fluence rate slightly different from the three others.

Figure 3 shows the expected values of the stochastic fluence rates and of the fluxes. The medium is correlated, $\alpha = 5k_0$, and $x = 30\%$. The average values decrease somewhat faster than $\exp(-k_0 z)$. They are almost the same as the fluence rate and the flux in a deterministic medium, $\Psi_0(z)$ and $F_0(z)$ [Eqs. (7)]. The stochastic averages and the deterministic values are even closer for a lower value of x or in a less-correlated medium. When the variation of the optical parameter is white noise, i.e., when $\alpha \to \infty$, they are identical [Eqs. (21)].

The standard deviation of the fluence rate in the same sample medium is shown in Fig. 4. When $x = 30\%$, the standard deviation is substantial compared with the average fluence rate, which in the figure is represented by $\Psi_0(x)$. For $\mu = \mu_s$, $\mu = g$, or $\mu = \mu_s'$, the standard de-

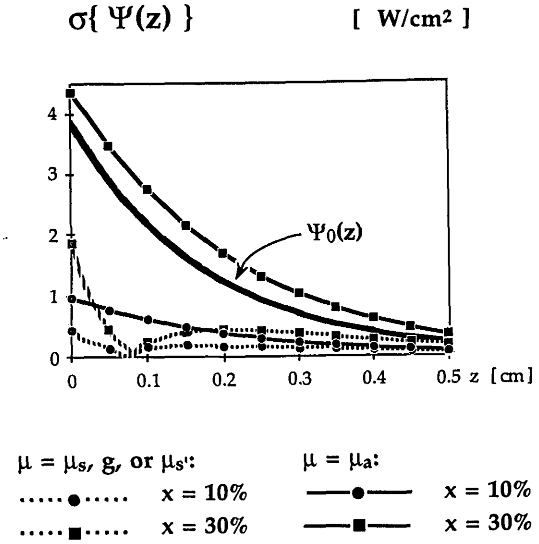

Fig. 4. Standard deviation of the light fluence rate. The distributions of the stochastic parameters of the medium are correlated: $\alpha = 5k_0$. Especially for higher x values, the standard deviation of the fluence rate is considerable compared with its average value, which is represented here by $\Psi_0(z)$.

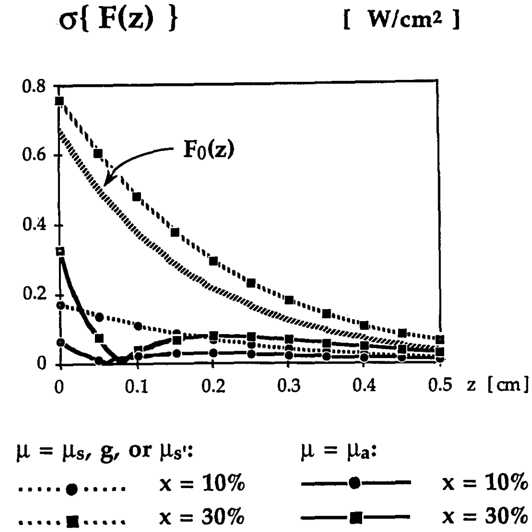

Fig. 5. Standard deviation of the light flux. The same graph as Fig. 4 but now for the light flux. The roles of the stochastic scattering and absorption parameters are reversed.

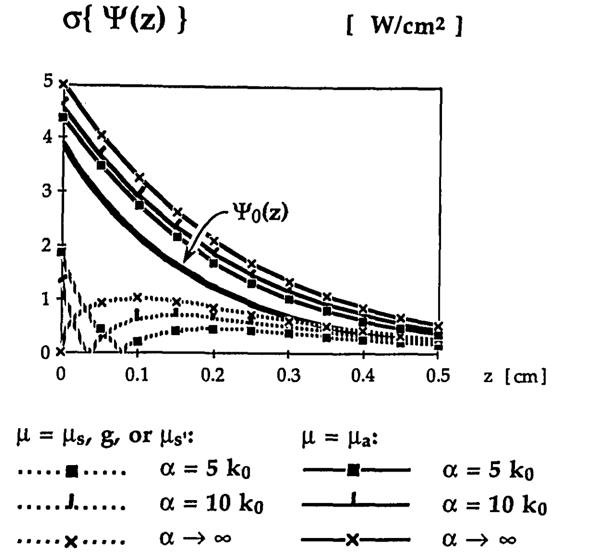

Fig. 6. Standard deviation of the light fluence rate for media with several correlation parameters α. The standard deviation ($x = 30\%$) of the fluence rate for a correlated medium ($\alpha = 5k_0$) does not differ too much from the one for a totally uncorrelated medium ($\alpha \rightarrow \infty$).

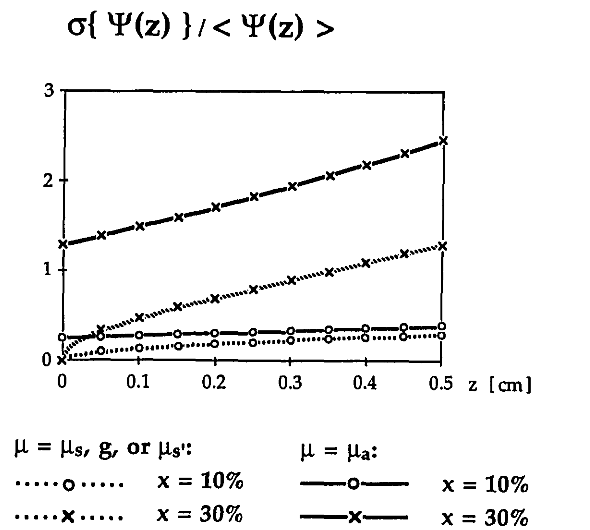

Fig. 7. Ratio of the standard deviation of light fluence rate and its average value. Here $\alpha \rightarrow \infty$, but for higher correlations the graphs are similar. Further into the medium, where the light has traversed more stochastic medium, the light fluence rate becomes relatively more noisy.

viation has an irregular minimum near the surface. This minimum is caused by one's having to minimize the difference with the boundary condition [expression (27)] instead of fulfilling the exact boundary condition. For the same reason, the magnitude of the standard deviation calculated here is only an indication of the real one. Parameters $\langle\Psi^2(0)\rangle$, $\langle\Psi(0)F(0)\rangle$, and $\langle F^2(0)\rangle$, which directly influence the magnitude of the standard deviation, are chosen as nearly as possible, but they are not the exact values. More important however is the change rate deeper in the tissue: the standard deviation of the fluence rate decreases nearly as $\exp(-k_0 z)$.

Figure 5 is the same graph as Fig. 4, except that in Fig. 5 the standard deviation of the flux is shown. Qualitatively, the behavior of $\sigma\{F(z)\}$ for $\mu = \mu_s$, $\mu = g$, or $\mu = \mu_s'$ is the same as the behavior of $\sigma\{\Psi(z)\}$ for $\mu = \mu_a$ and vice versa.

In Fig. 6 the stochastic optical parameters have the same relative standard deviation ($x = 30\%$), but the correlation α is varied. In a rather correlated medium ($\alpha = 5k_0$), the standard deviation of the fluence rate does not differ greatly from the standard deviation in a totally uncorrelated medium ($\alpha \rightarrow \infty$). Both the average value of the fluence rate and the standard deviation vanish for $z \rightarrow \infty$. The exponential decrease of $\sigma\{\Psi(z)\}$ is, however, slower than the exponential decrease of $\langle\Psi(z)\rangle$. As shown in Fig. 7, the ratio $\sigma\{\Psi(z)\}/\langle\Psi(z)\rangle$ increases deeper in the tissue[22]: the fluence rate becomes more noisy further from the light source.

DISCUSSION AND CONCLUSIONS

In this paper a method is derived for calculating the expected value and the standard deviation of the light fluence rate in a medium with stochastic optical parameters. For the stochastic optical parameters, no probability distribution in each point is assumed; only a current postulation is made about the correlation between two distributions at different points.

In calculating the statistical moments of the light fluence rate and the light flux, we disregarded higher-order correlations. This is a valid assumption if the variations of the optical parameter are not too large and if the medium is not too correlated, which is usually so for a biological tissue. Still, the results of this paper are primarily qualitative and give more an indication of the statistical moments of the fluence rate than their exact values. It is difficult to support the results with measurements. The method could be independently verified, though, by performance of perturbation calculations with the Monte Carlo method.[23]

The stochastic moments of the fluence rate do not depend strongly on the correlation between the layers. Most often, it will be sufficient to consider only an uncorrelated medium, i.e., a medium with white noise in one of the optical parameters.

In other stochastic processes, the expected value of the process can be significantly different from the value calculated with the average values of the parameters (see, for example, Ref. 2). In our problem, the fluence rate of light in a highly scattering medium, the differences are small. Calculations with the average values of the optical parameters can be considered unbiased.

When the standard deviation of an optical parameter is ~30%, the standard deviation of the fluence rate is considerable: it is of the same order of magnitude as the average value of the fluence rate. The standard deviation decreases further from the light source but more slowly than the average fluence rate. Relative to the average value, the standard deviation increases, with a rate depending on parameter x. The fluence rate is more noisy further from the source.

The geometry in this paper varies only in one dimension. In real, three-dimensional geometries, the optical parameters vary also in the two lateral directions, and so the fluence rate varies also laterally. Since light is also scattered into lateral directions, its standard deviation is thus higher in a three-dimensional geometry. In measurements, part of this extra variation will not be noticed, though, because of the spatial resolution of the measuring device, which makes a measured value an average value over a certain surface area.

To measure the total reflection of a tissue medium, one can use an integrating sphere. In those measurements, the light radiance at the $z = 0$ surface is sampled. Therefore the expected value of the stochastic total reflection is almost the same as the deterministic total reflection. The standard deviation of the total reflection is of the same order of magnitude as the average total reflection (if $x = 30\%$). For a medium of finite thickness the total transmission can be measured. The expected value of the total transmission is also almost the deterministic value, but the relative standard deviation is higher than that of the total reflection. For thicker samples, the reflection measurements are less noisy than the transmission measurements.

In a medium with varying optical parameters, measurements of the fluence rate, of the flux, or of the total reflection or transmission are noisy, not only because of noise in the measuring apparatus or because of the specimen-to-specimen variation but also because of small-scale inhomogeneities within one sample. The last cause is more important further away from the light source. One can estimate the average values with some accuracy by repeating the measurements at different spots of the sample.

Often a medium has to be irradiated, while each spot in the medium should not get more than a certain absorbed light dose or not less than a certain dose. In order to be sure (with a certain chance) that the dose limit is not overstepped, not only the deterministic fluence rates but also the statistical confidence intervals for the fluence rates have to be estimated.

In many applications the light dose is not so important as the subsequent heat dose. The variations in heat dose will be much less than the variations in light dose, because heat diffusion spatially averages the light dose. However, the heat dose is calculated from the light dose by means of a nonlinear integral equation. Spikes in the light dose give spikes in the initial heat distribution and could thus show up as (smaller) spikes in the heat dose. Dose calculations should take into account that some spots can experience much less and others much more heat damage than was estimated with the deterministic calculations.

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12. Only in the figure is the extra assumption made that μ_1 at one depth is distributed Gaussian: $\mu_1(z) \sim N(0, \alpha\sigma^2)$. In that case the μ_1 distribution is a Markov process in z and can be sampled easily.[11]
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17. Incidentally, from relation (14) it cannot be concluded that the variance of the fluence rate or of the flux is assumed to vanish at $z = 0$; the stochastic process modifies $\langle\Psi(0)\rangle$, $\langle F(0)\rangle$, and the higher-order moments.
18. If $\Psi(z)$ vanishes for $z \to \infty$, then surely also $\langle\Psi(z)\rangle$ must vanish for $z \to \infty$. At $z = 0$, a linear relation exists between $\Psi(0)$ and $F(0)$: [Eq. (10c)]. So Eq. (20a) can be derived.
19. The two largest poles of $L\{\langle\Psi(z)\rangle\}$ [Eqs. (17)] have values of approximately $-k_0$ and $+k_0$, respectively. The negative pole is the important pole; the positive pole is unwanted because of boundary condition (20b). The three largest poles of $L\{\langle\Psi^2(z)\rangle\}$ are combinations of those two poles of $L\{\langle\Psi(z)\rangle\}$. Both combinations with the positive pole, s_5 and s_6, also turn out to be unwanted (see Note 21 below). The most important pole in $L\{\langle\Psi^2(z)\rangle\}$ is approximately the square of the most important pole of $L\{\langle\Psi(z)\rangle\}$: $\sim -2k_0$, but not exactly (see Note 22 below).
20. $F(0)$ from Eq. (10c) is substituted into $\langle\Psi(0)F(0)\rangle$ and into $\langle F^2(0)\rangle$. Thus the two are expressed in $\langle\Psi(0)\rangle$ and $\langle\Psi^2(0)\rangle$ only. A combination of the results is (an exact) boundary condition at $z = 0$: $\langle\Psi^2(0)\rangle + 4A\langle\Psi(0)F(0)\rangle + 4A^2\langle F^2(0)\rangle = F_i^2$.
21. In the expression for either $\langle\Psi^2(z)\rangle$ or $\langle F^2(z)\rangle$, the factor before the term $\exp(-s_5 z)$ is not a positive value. A negative factor there would result in a negative $\langle\Psi^2(z)\rangle$ or $\langle F^2(z)\rangle$ deeper in the medium, and one of the conditions (26) would not be met.
22. The largest pole of $L\{\langle\Psi(z)\rangle\}$ that is left after the boundary conditions have been applied is smaller than half of the largest pole of $L\{\langle\Psi^2(z)\rangle\}$ that is left after its boundary conditions have been applied. Thus $\sigma\{\Psi(z)\}/\langle\Psi(z)\rangle$ increases for larger depths.
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